Longest Increasing Subsequence

2013 IOI Camp 1

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Introduction

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But how to do that?

First Attempt

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But what is its efficiency?
$O(2^n)$! Yikes!

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Store the length of the longest increasing subsequence ending on that point. This can also be used to reconstruct the subsequence.

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1	2	2	3	2	3	4	5	2	6

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for i from 1 to n do
  best = 0
  for j from 1 to i-1 do
    if s[j] < s[i] and dp[j] > best then
      best = dp[j]
  dp[i] = best + 1
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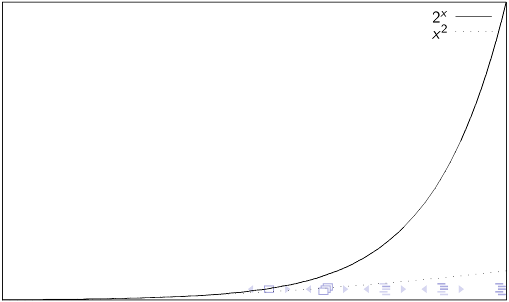

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We also don't have a very easy way of finding the sequence (one exists but it can be bettered)

A different DP is needed.

Third Attempt

If we have a choice amongst previous elements when building our LIS, we might as well take the smallest. This leads to the DP:

- Let $m[j]$ store the position k of the smallest $s[k]$ such that there is a increasing subsequence of length j ending on $s[k]$.
- Let $p[i]$ store the predecessor of $s[i]$ in the longest increasing subsequence ending on $s[i]$.

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It is important to note that $s[m[1]], s[m[2]], \ldots, s[m[L]]$ is nondecreasing. This is true, as if there is a increasing subsequence of length i ending at $s[m[i]]$, then there is also a increasing subsequence of length $i - 1$ ending at a smaller value, i.e. the all-but-one of that sequence.

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Then we can build this up as follows:

Psudocode

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L = 0
for i = 1 to n do
  binary search for the largest positive j  L
    such that s[m[j]] < s[i] (or set j = 0 if no such value
  P[i] = m[j]
  if j == L or s[i] < s[m[j+1]]:
    m[j+1] = i
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This has $O(n \log n)$ which is good enough for most cases.