Intro to Graph Theory

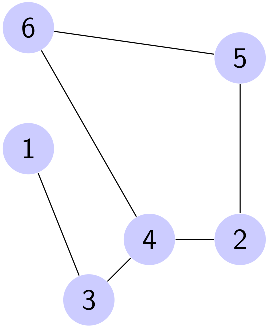
2014 IOI Camp 1

Robert Spencer

December 11, 2013

Introduction

This is a graph:



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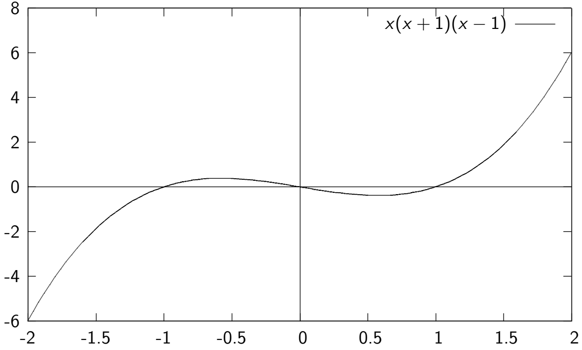

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A graph is a collection of *nodes* connected by *edges* which may or may not be *directed* and/or *weighted*

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Examples of graphs:

- A computer network (non-directed, non-weighted)
- A road map (non-directed, weighted)
- Winners in a chess tournament (directed, non-weighted)
- Payments in an economy (weighted, directed)

Paths and Cycles

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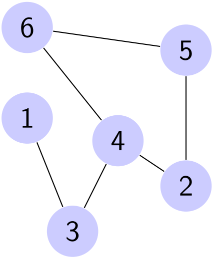
Can you find a path and a cycle?



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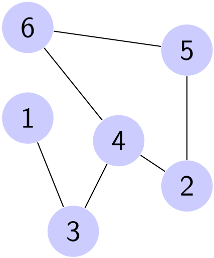

Example Answer:
Path: 1-3-4-2
Cycle: 4-2-5-6-4

Trees

Definition

A tree is an un-directed complete graph with no cycles.

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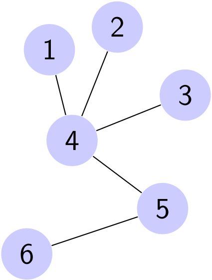
Example



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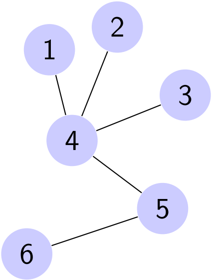

Theorem

A tree of n vertices has $n-1$ edges.

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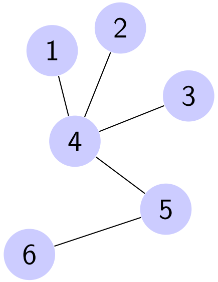

Theorem

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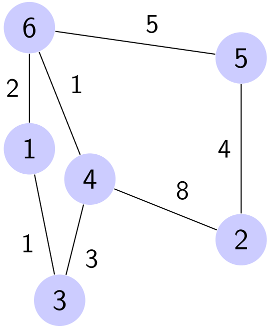
Proof.

Induction. Start with one vertex, and add subsequent ones. □

Trees

Weights are placed on edges, and can represent anything (lengths, costs, etc.)



How do we represent a graph?

Graph Representations

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Lists of Neighbours

```
[(3,1),(6,2)]
[(4,8),(5,4)]
[(1,1),(4,3)]
[(2,8),(3,3),(6,1)]
[(2,4),(6,5)]
[(5,5)]
```

Memory $O(E)$

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Memory $O(E)$

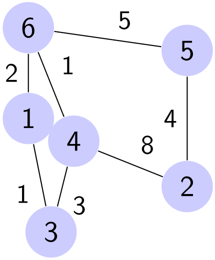
Adjacency Matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 8 & 4 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 8 & 3 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{pmatrix}$$

Memory $O(N^2)$

Traversal

Sometimes we want to visit all the nodes in a graph in a particular order. For example to search for a path/destination



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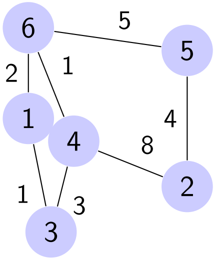

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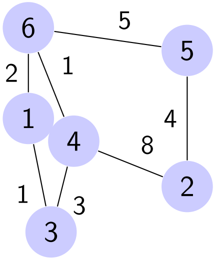

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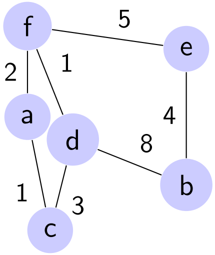
Often this is used to find the shortest route between two or more nodes.

Depth First Search

Depth First Search (DFS) visits the nodes as far as it can before backtracking (without visiting nodes more than once).

Sample Graph:



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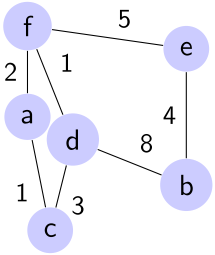

Psudocode:

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def DFS(currNode, finalNode)
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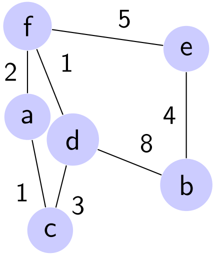

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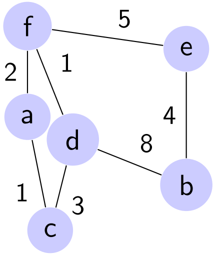

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a-c-d-b-e-f-f-e-b-f-d-c-b-e

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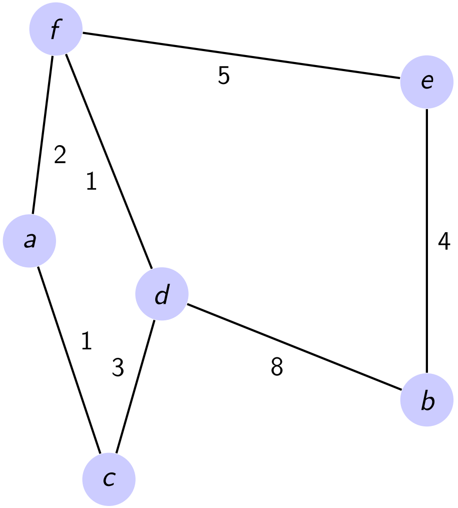

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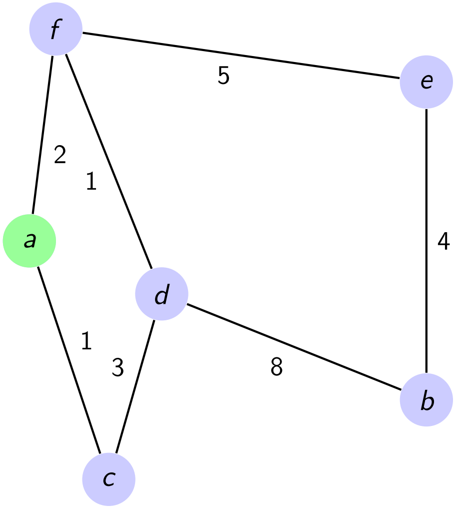

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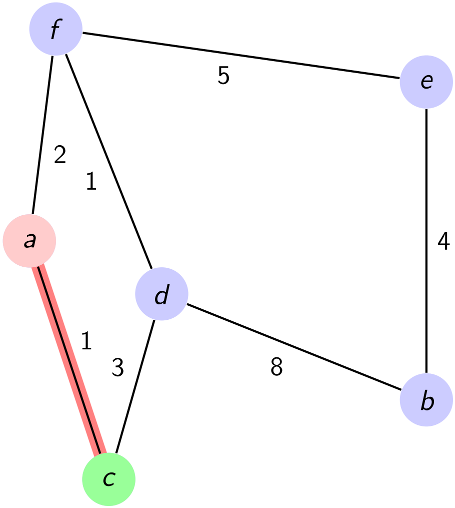

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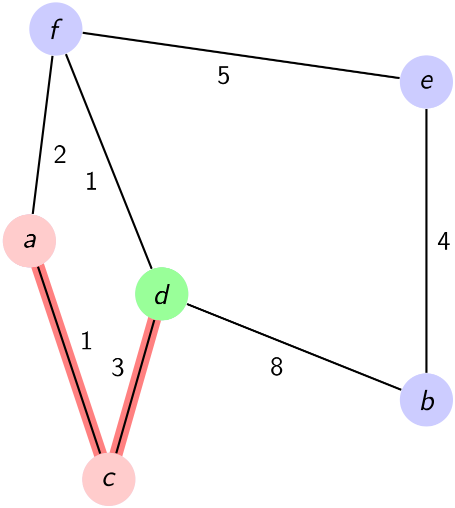

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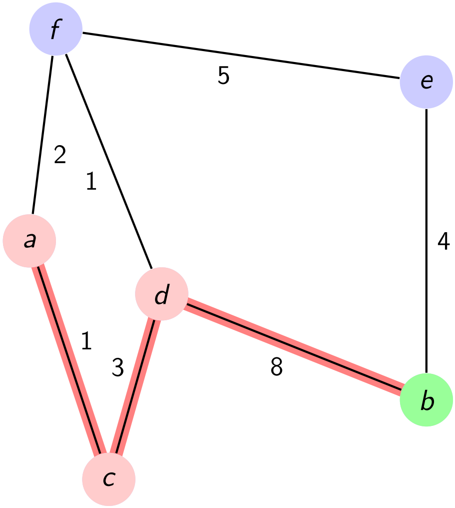

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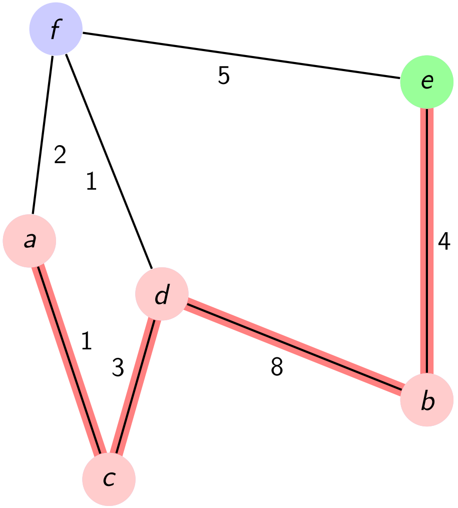

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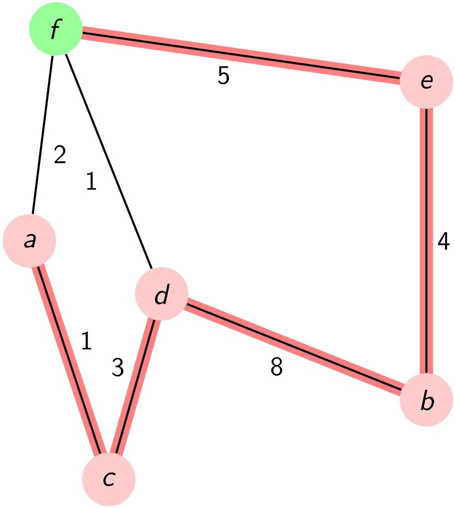

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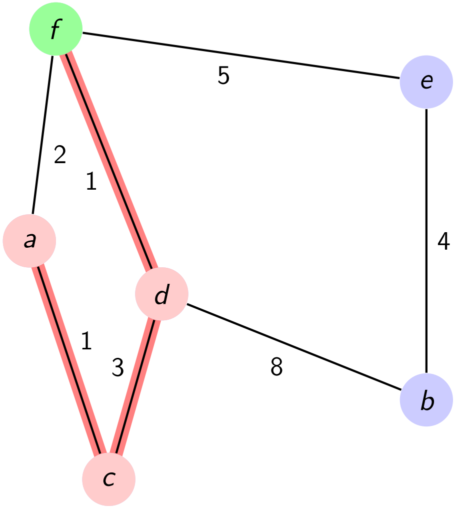

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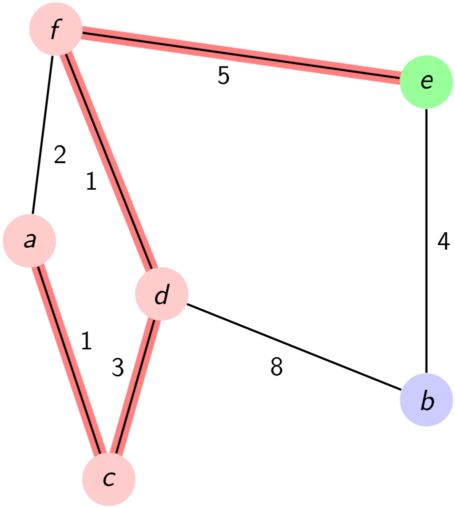

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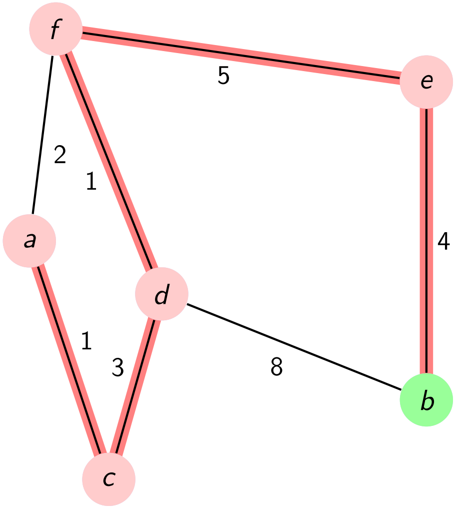

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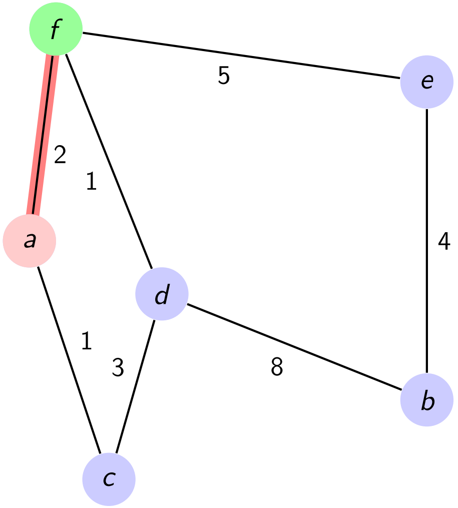

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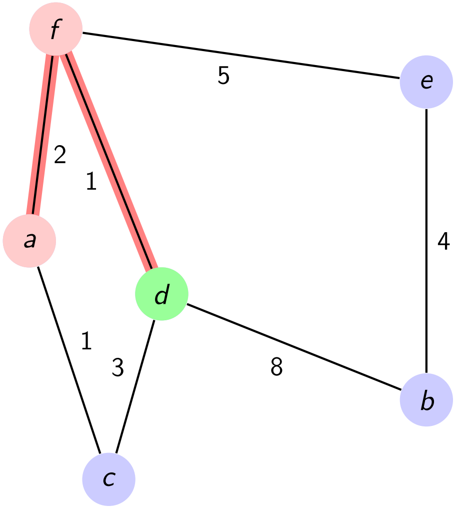

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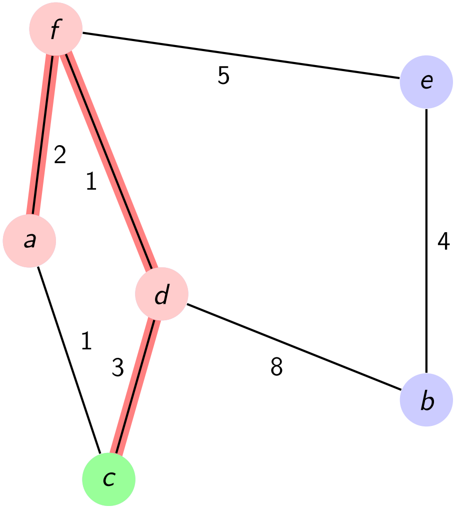

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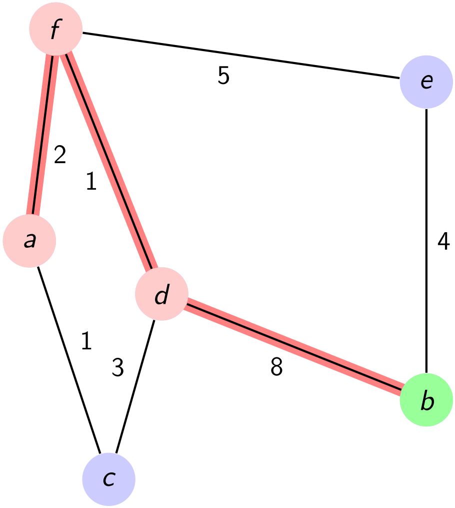

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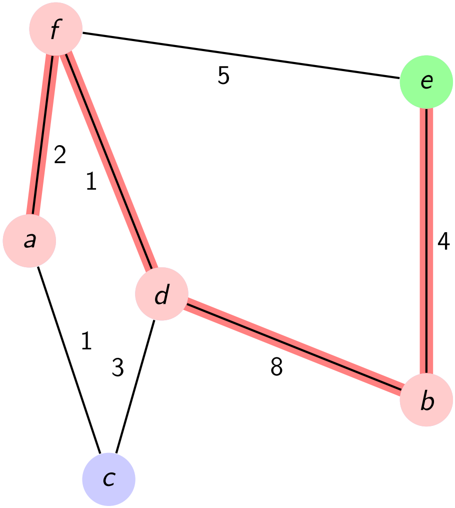

Breadth First Search

Breadth First Search (DFS) visits the nodes "in parallel" without backtracking.

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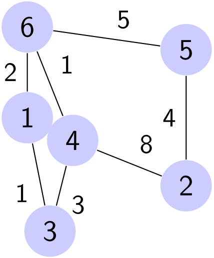

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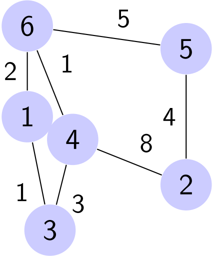

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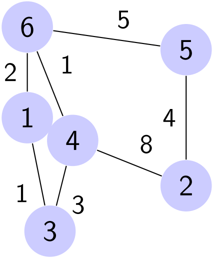

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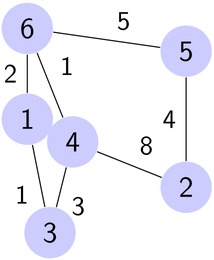

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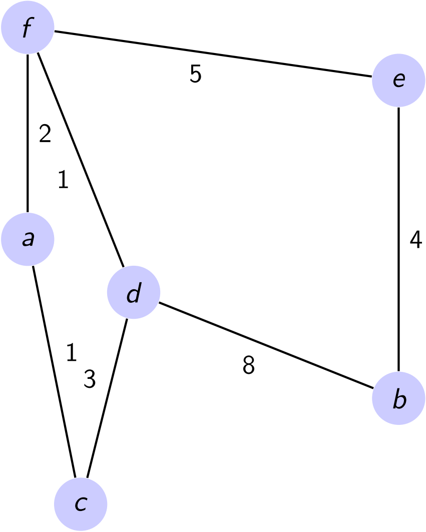

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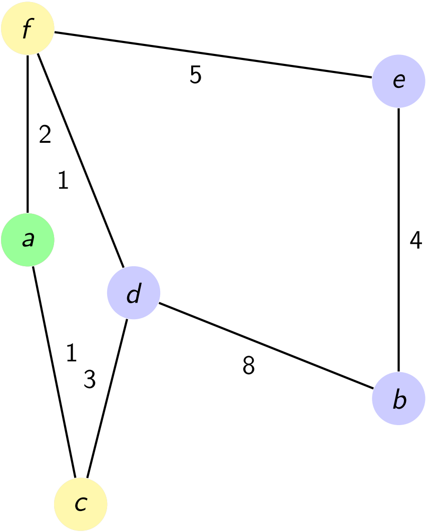

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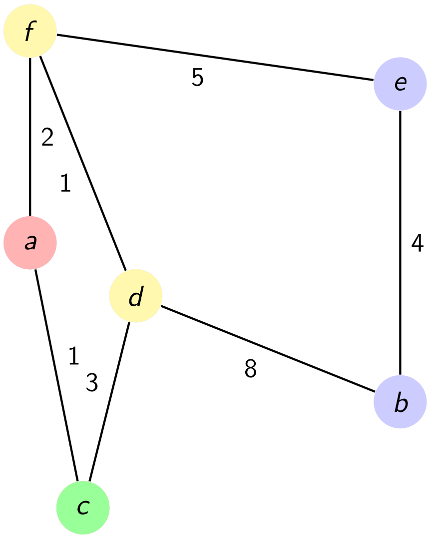

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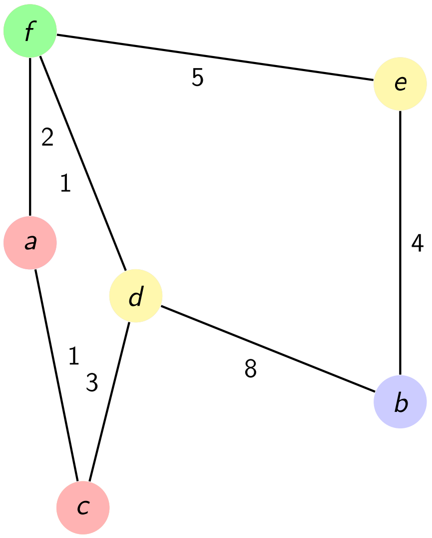

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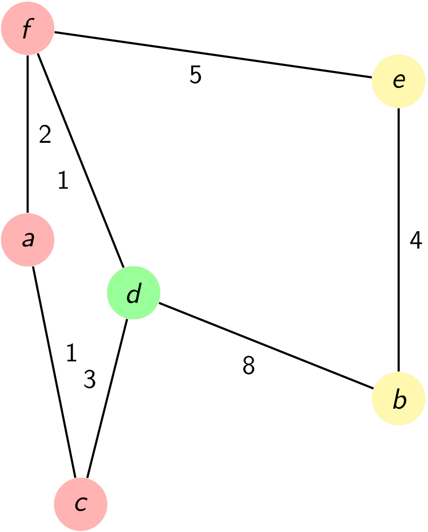

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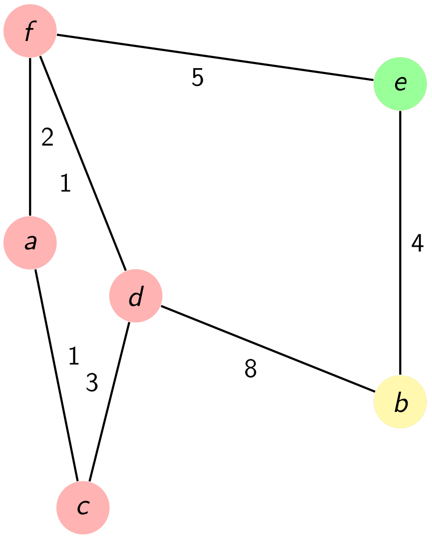

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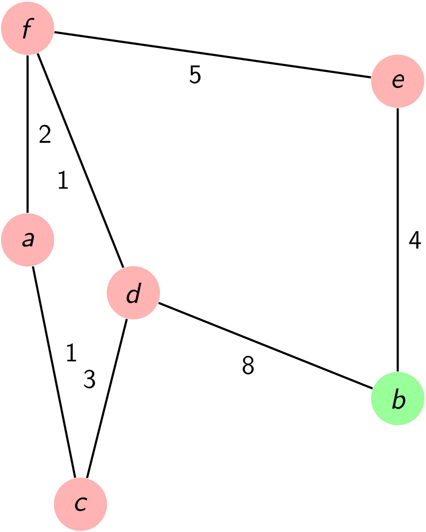

Dijkstra's Algorithm

Dijkstra's Algorithm finds the shortest distance from one node to all others. It is basically a BFS with a priority queue.

Psudocode:

```
set all distances INF
add (0, startNode) to queue
while queue not empty do
  currDists,currNode = queue.pop
  distances[currNode] = currDist
  for neighbour,distance in adjacent[currNode] do
    possNewDist = distances[currNode] + distance
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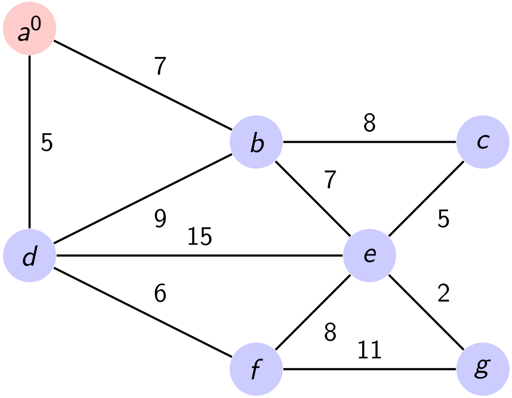
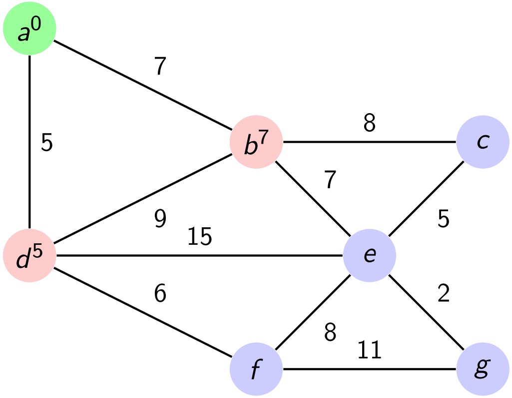


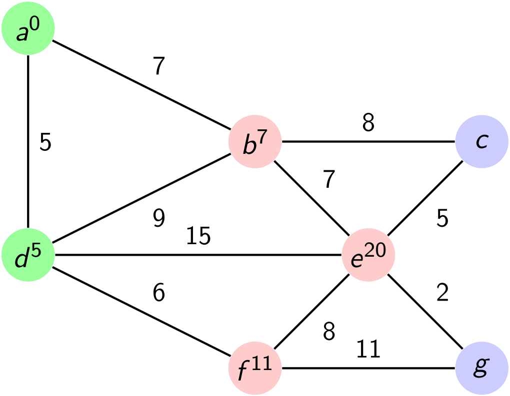
Queue: $\{(a, 0)\}$

Dijkstra Example



Queue: $\{(d,5),(b,7)\}$

Dijkstra Example



Queue: $\{(b,7),(f,11),(e,20)\}$

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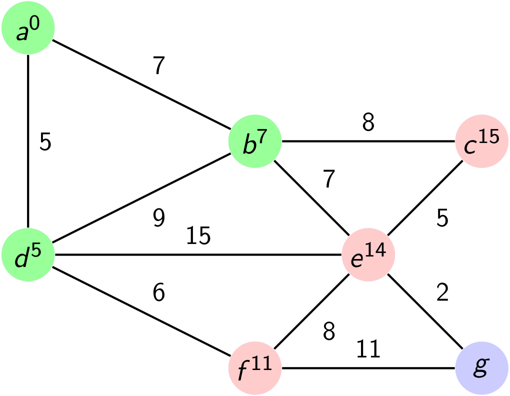
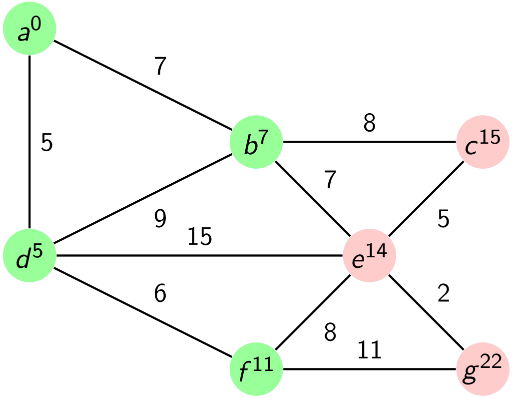


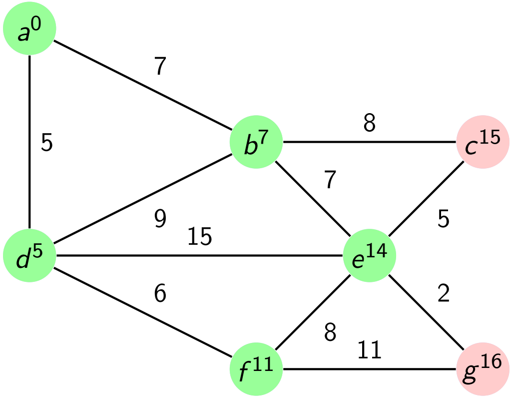
Queue: $\{(f, 11), (e, 14), (c, 15)\}$

Dijkstra Example



Queue: $\{(e, 14), (c, 15), (g, 22)\}$

Dijkstra Example



Queue: $\{(c, 15), (g, 16)\}$

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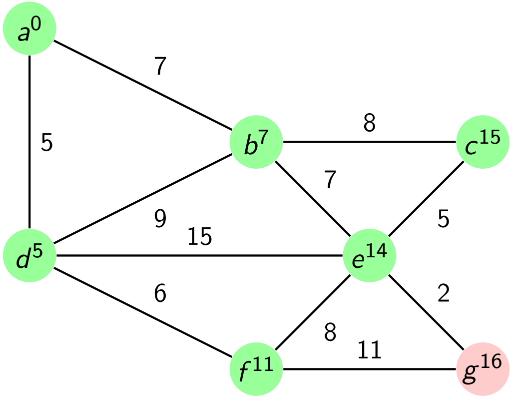

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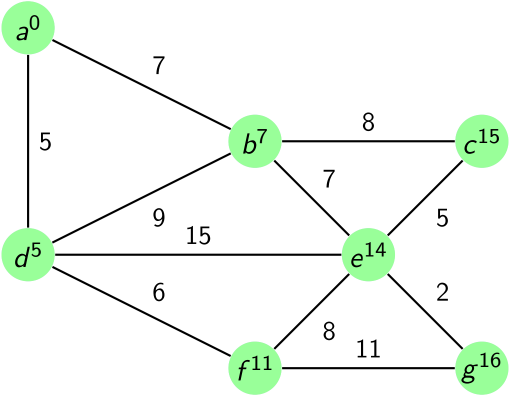
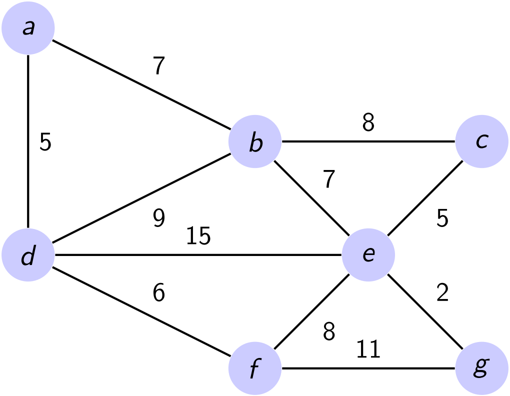


Queue: {}

Minimum Spanning Tree

Definition

A *minimum spanning tree* is a subset of edges in a weighted undirected graph such that the edges form a tree containing all the nodes, and the sum of the weights of the tree is minimal.



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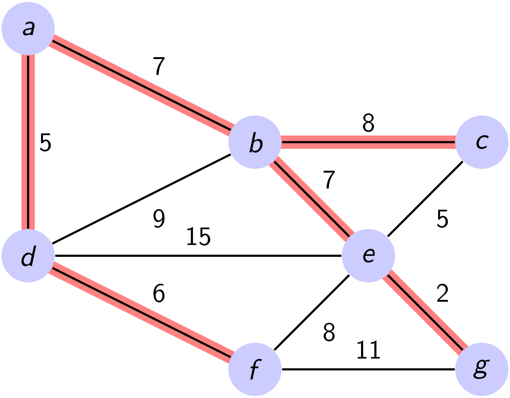

Prim's Algorithm finds the minimum spanning tree from a given graph.

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Algorithm

- Set all vertices to "not in the tree" except a starting vertex.
- While there are vertices not in the tree, add the vertex which is connected to the tree by the shortest edge to the tree.

Prim's Algorithm

Prim's Algorithm finds the minimum spanning tree from a given graph.

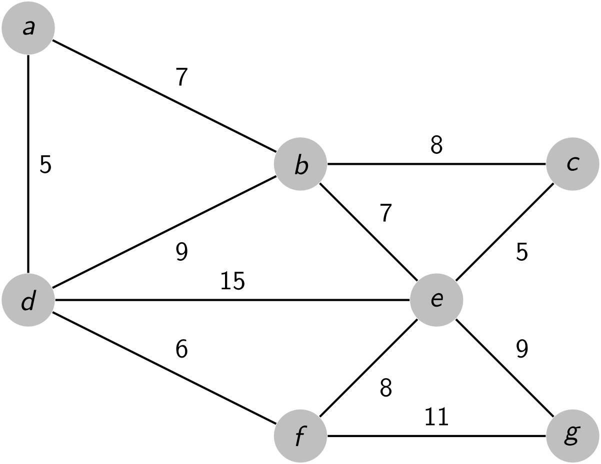
Algorithm

- Set all vertices to "not in the tree" except a starting vertex.
- While there are vertices not in the tree, add the vertex which is connected to the tree by the shortest edge to the tree.

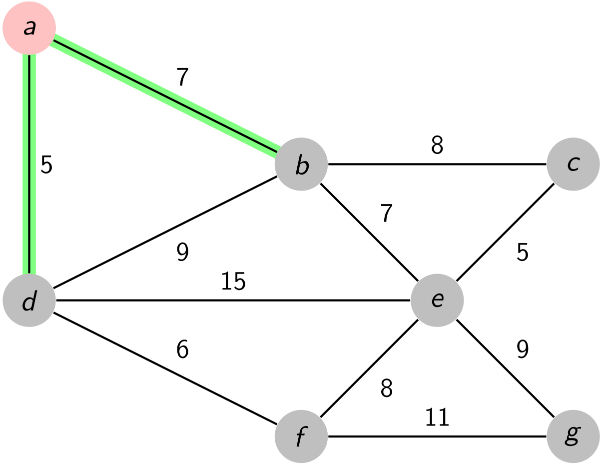
Technical notes

- Keep a priority queue of edges. Each step pull off an edge, check if it joins a new vertex. If it does, add all the edges from that vertex to the queue.
- Runs in $O(E \log V)$ with a binary heap as priority queue.

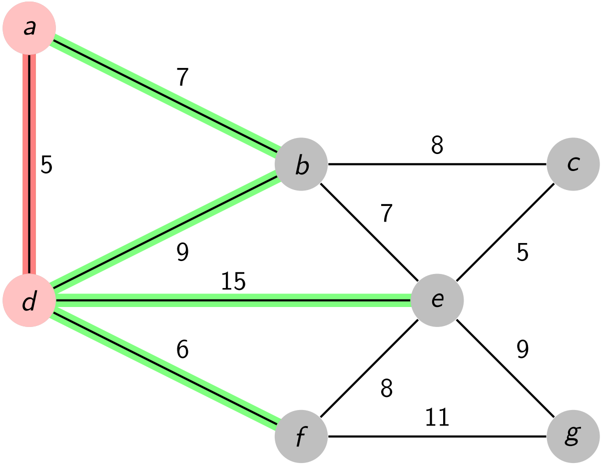
Prim's Algorithm Example



Prim's Algorithm Example



Prim's Algorithm Example



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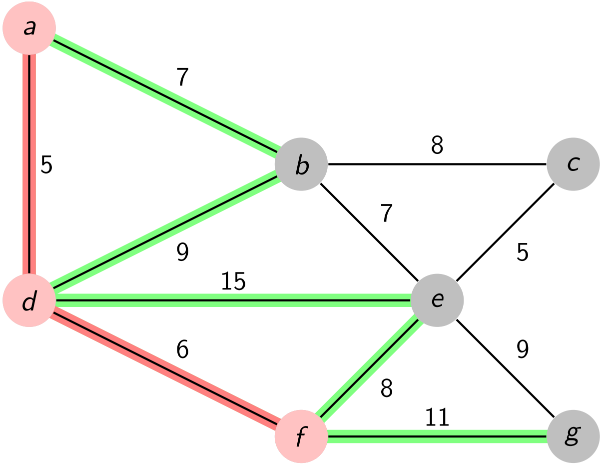

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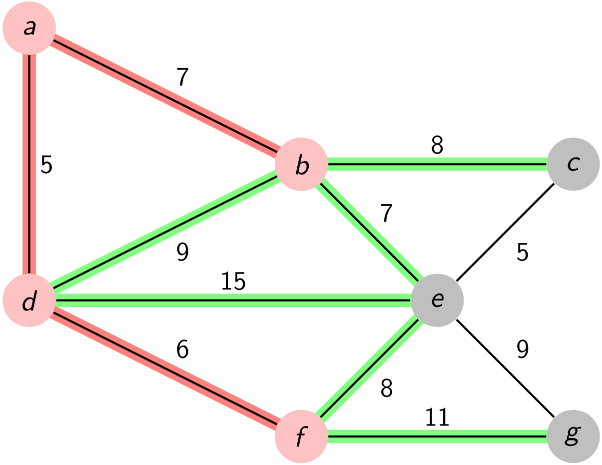

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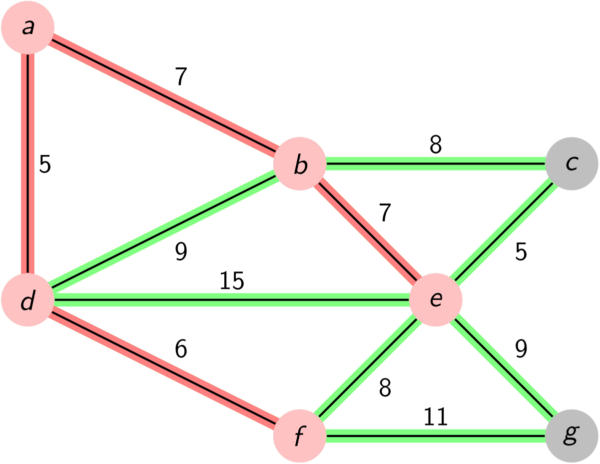

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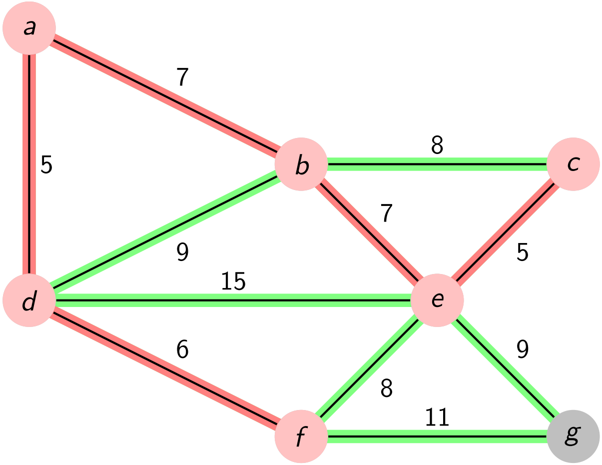

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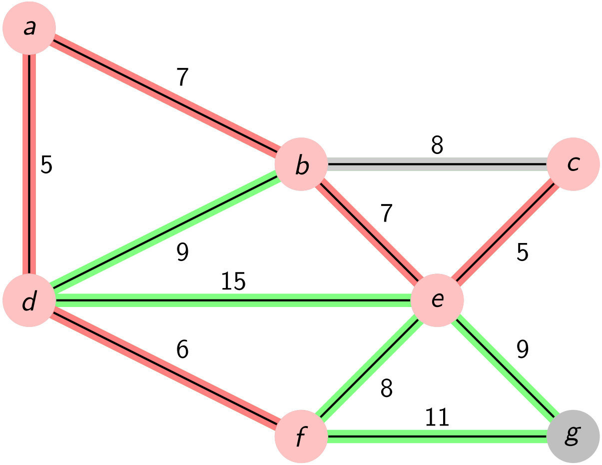

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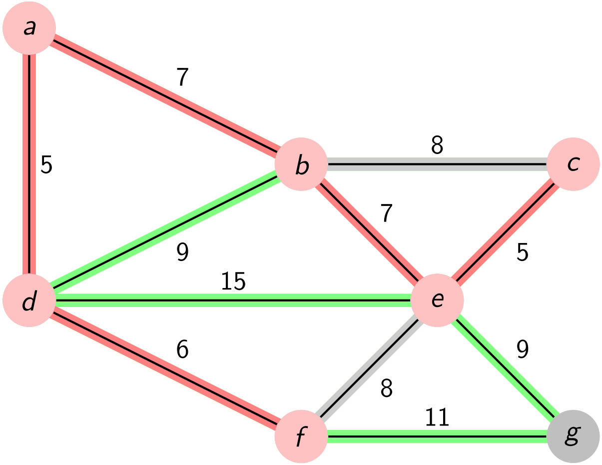

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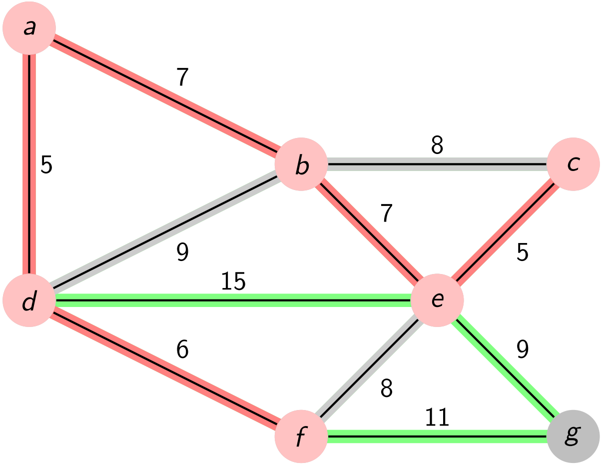

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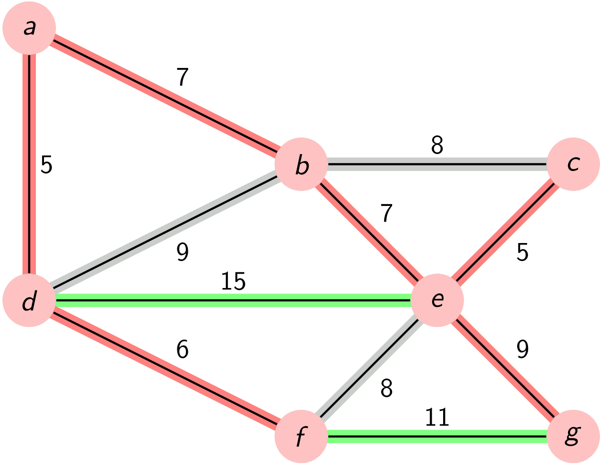

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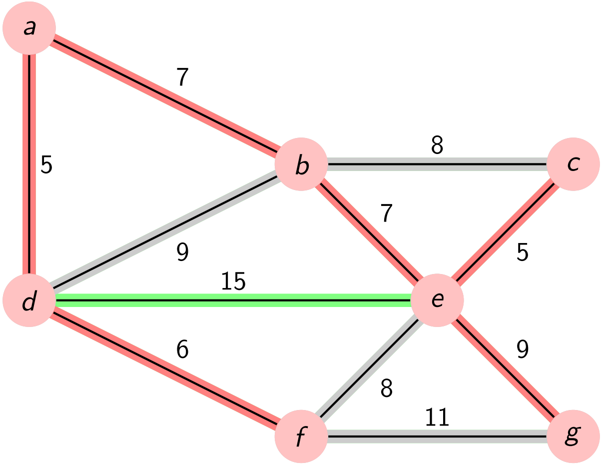

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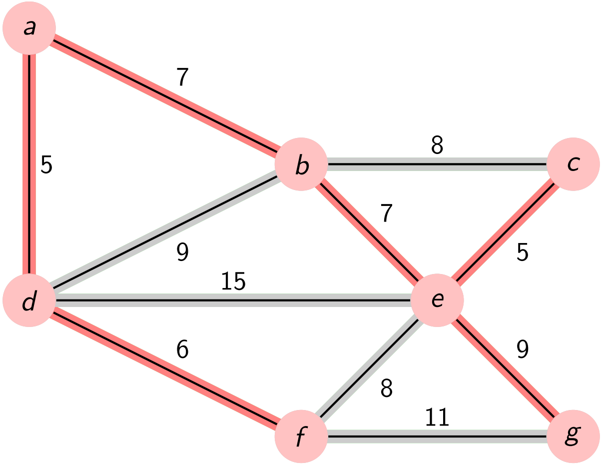

Kruskal's Algorithm

Kruskal's Algorithm is the "dual" of Prim's. It also finds the minimum spanning tree.

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Algorithm:

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Algorithm:

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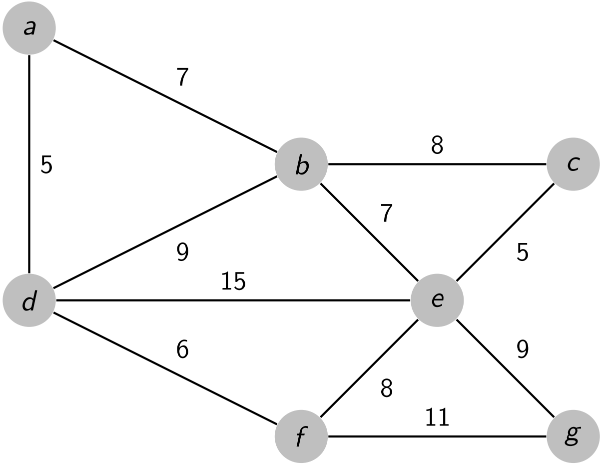
Technical Notes:

- Use "union-find" to hold the different trees.
- Complexity $O(E \log E)$

Kruskal's Algorithm Example



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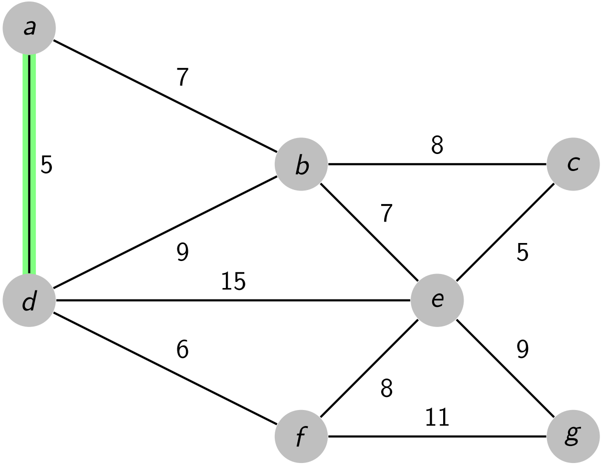

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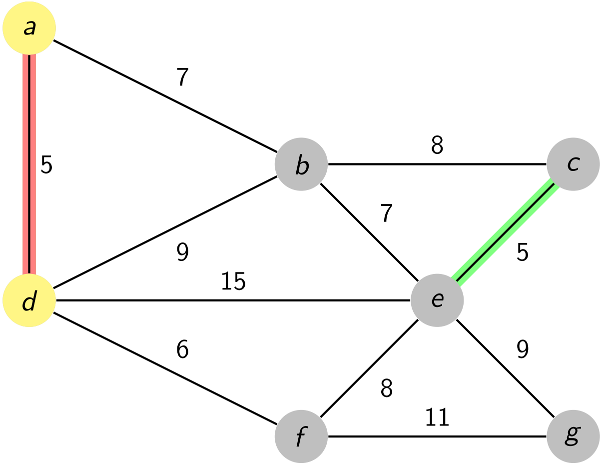

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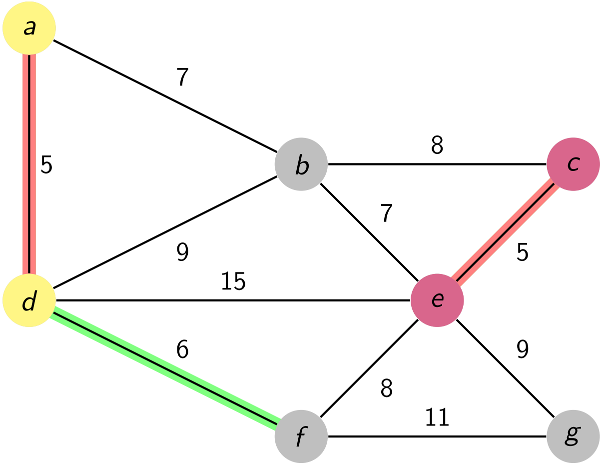

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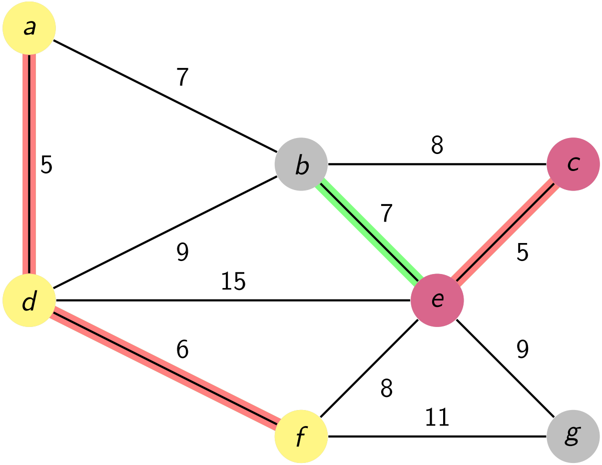

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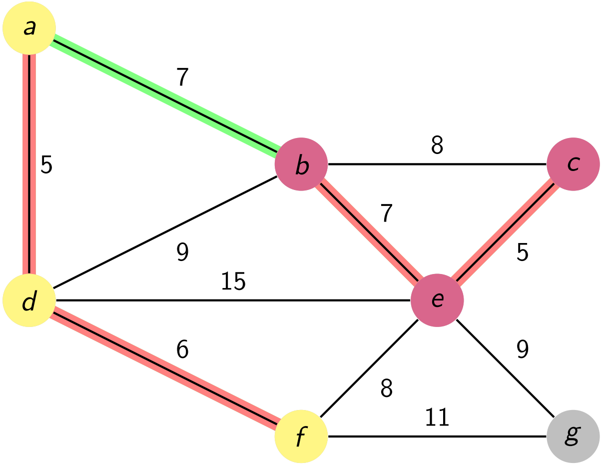

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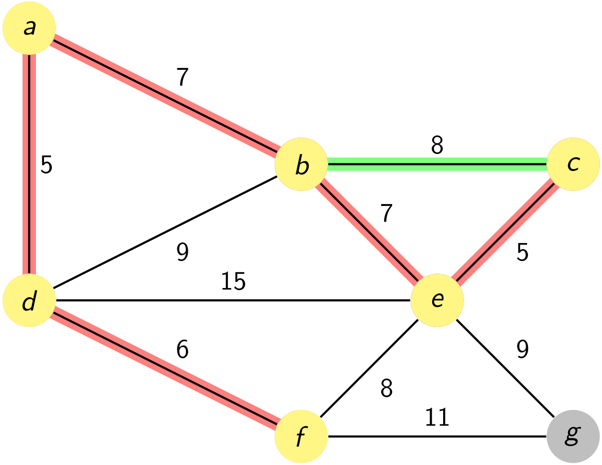

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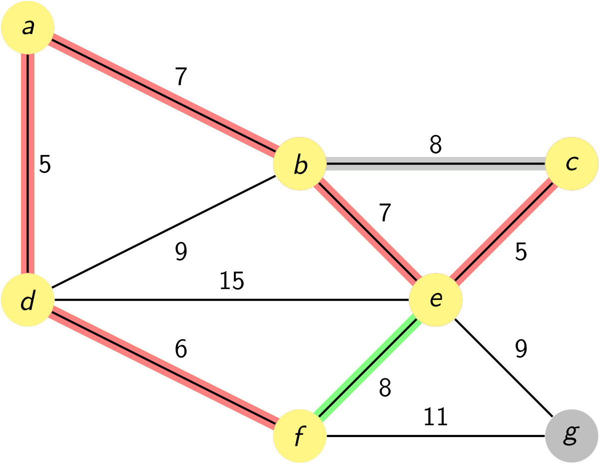

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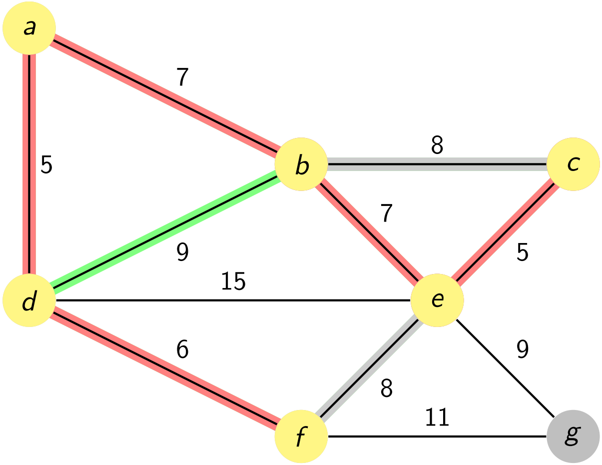

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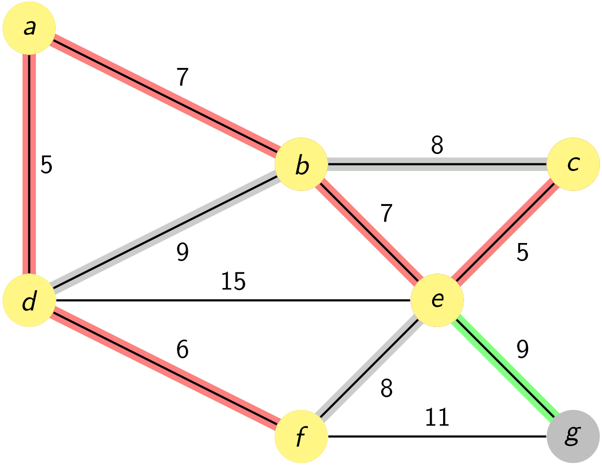

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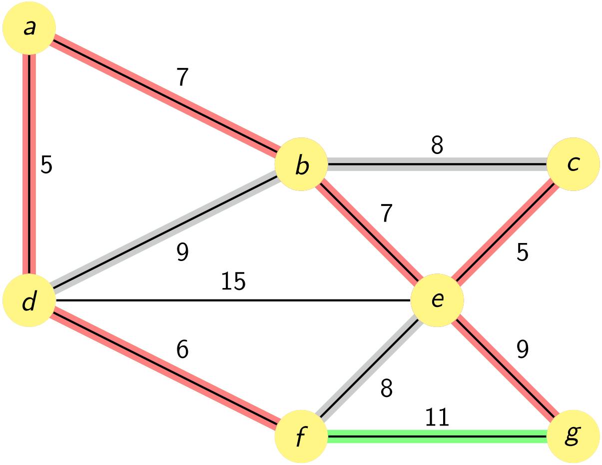

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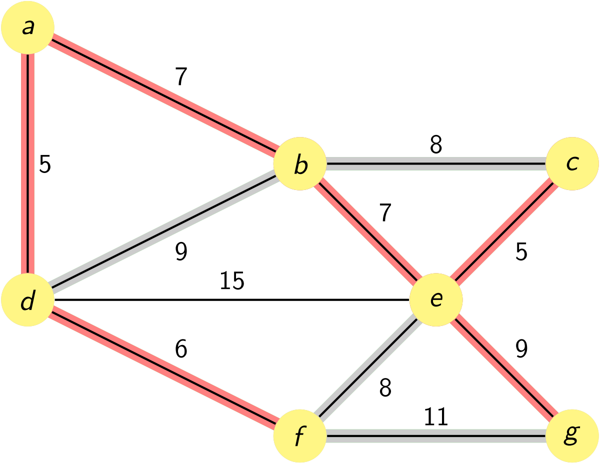

Kruskal's Algorithm Example