Line Sweep Algorithms

Schalk-Willem Krüger – 2009 Training Camp 1

```
presentation.start();
```

Closest Pair Algorithm

The problem

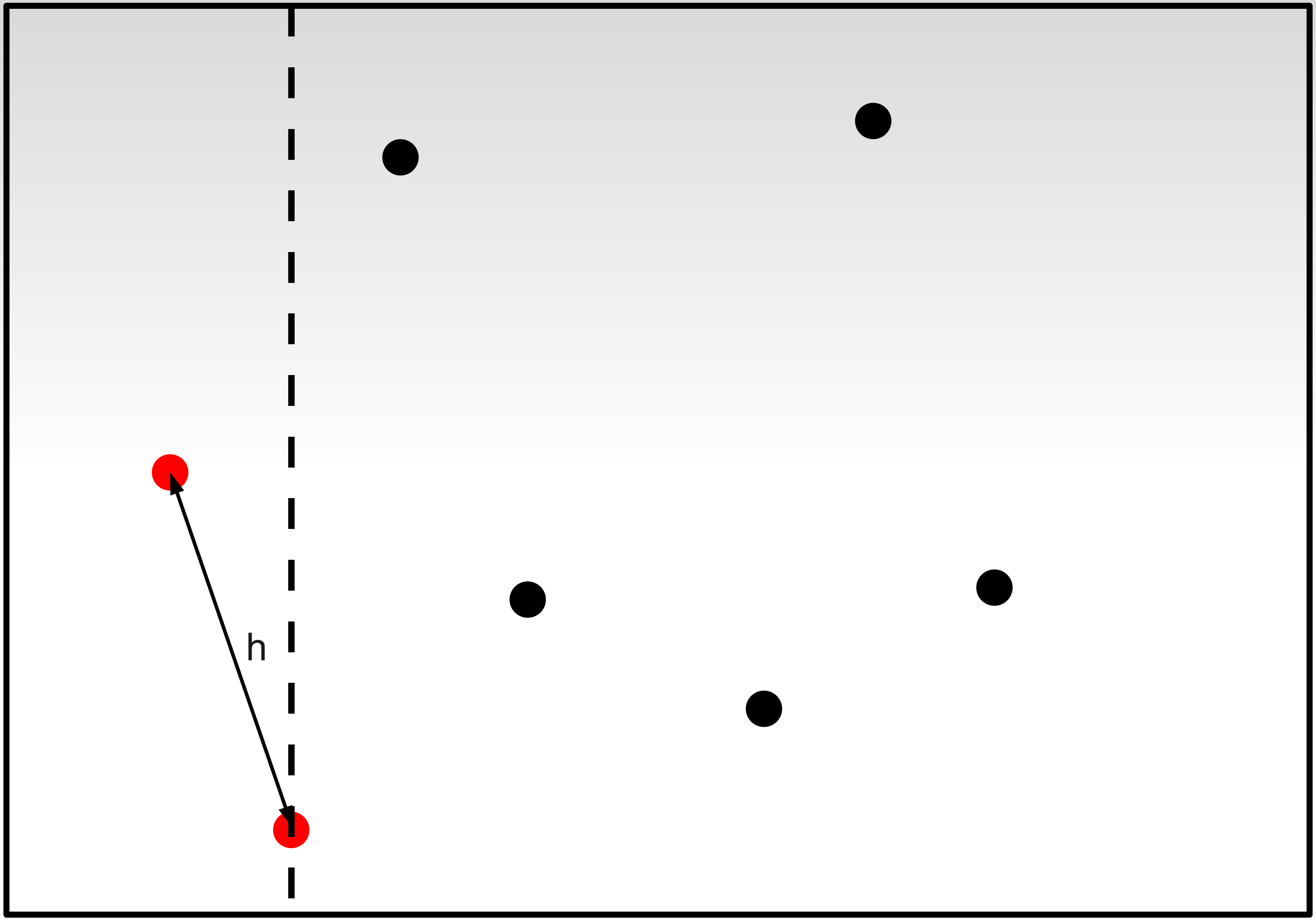
2008 Training Camp 2: Good Neighbours

<u>The problem:</u>

- Bruce wants to visit his friends on weekends.
- The friends are scattered around (each at a unique location).
- Find the two friends that live closest to each other
- Maximum of 1 000 000 friends

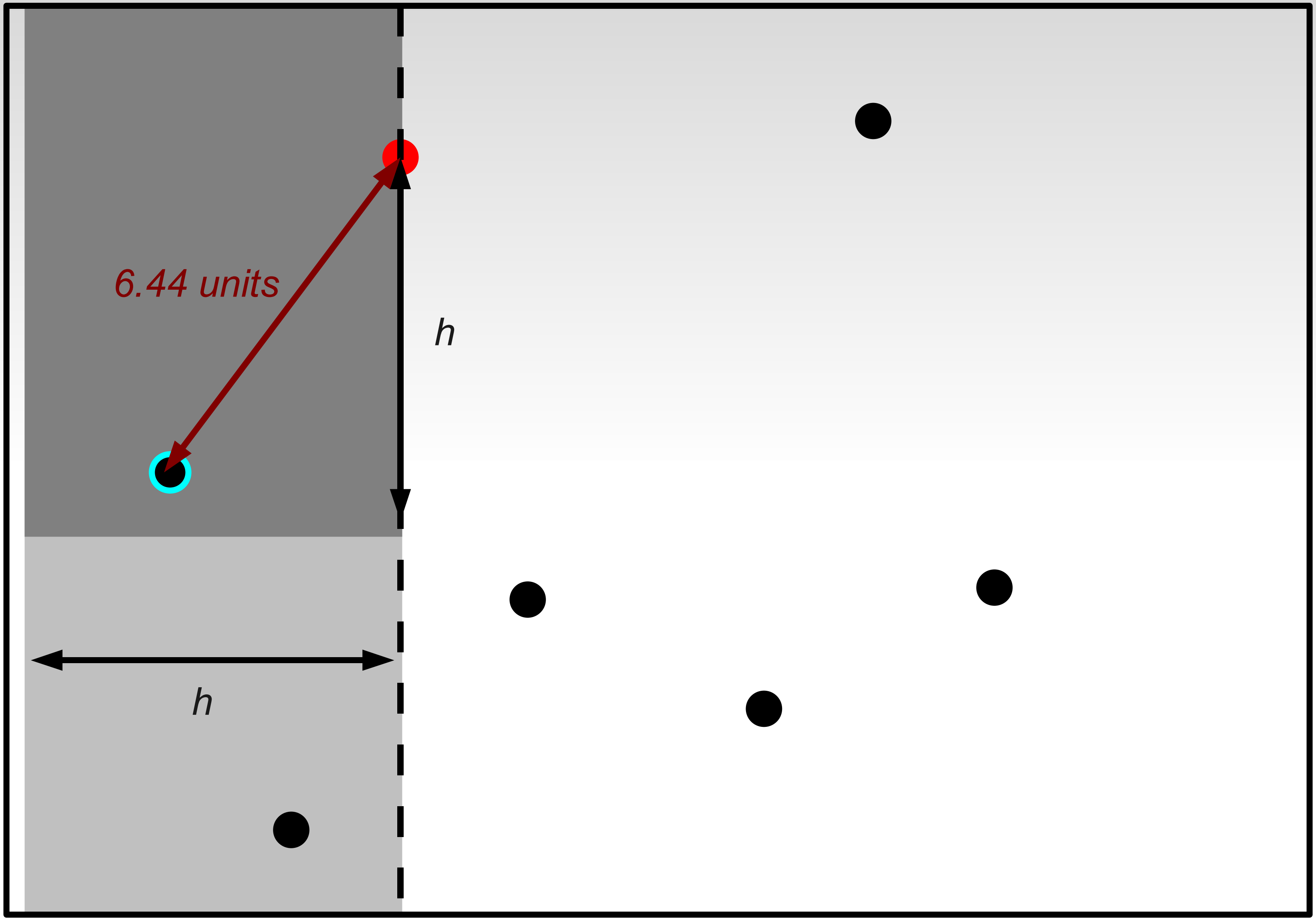
<u>Brute force:</u> $O(N^2)$ – too slow

Closest Pair Algorithm



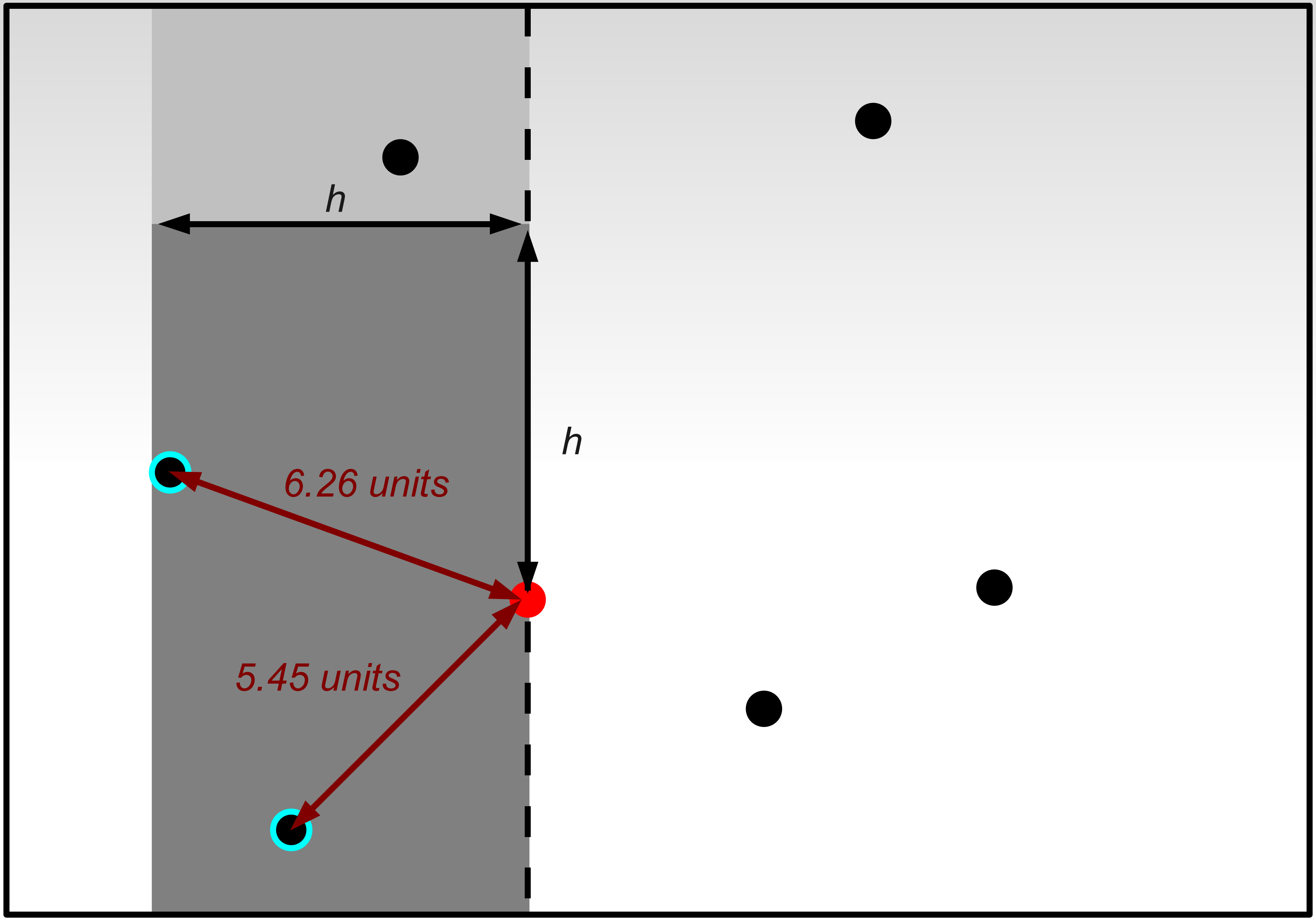
Initialize *h* (shortest distance found so far) with the distance between the first two points.
h = Euclidian distance between point 1 and 2 = 6.23 units

Closest Pair Algorithm



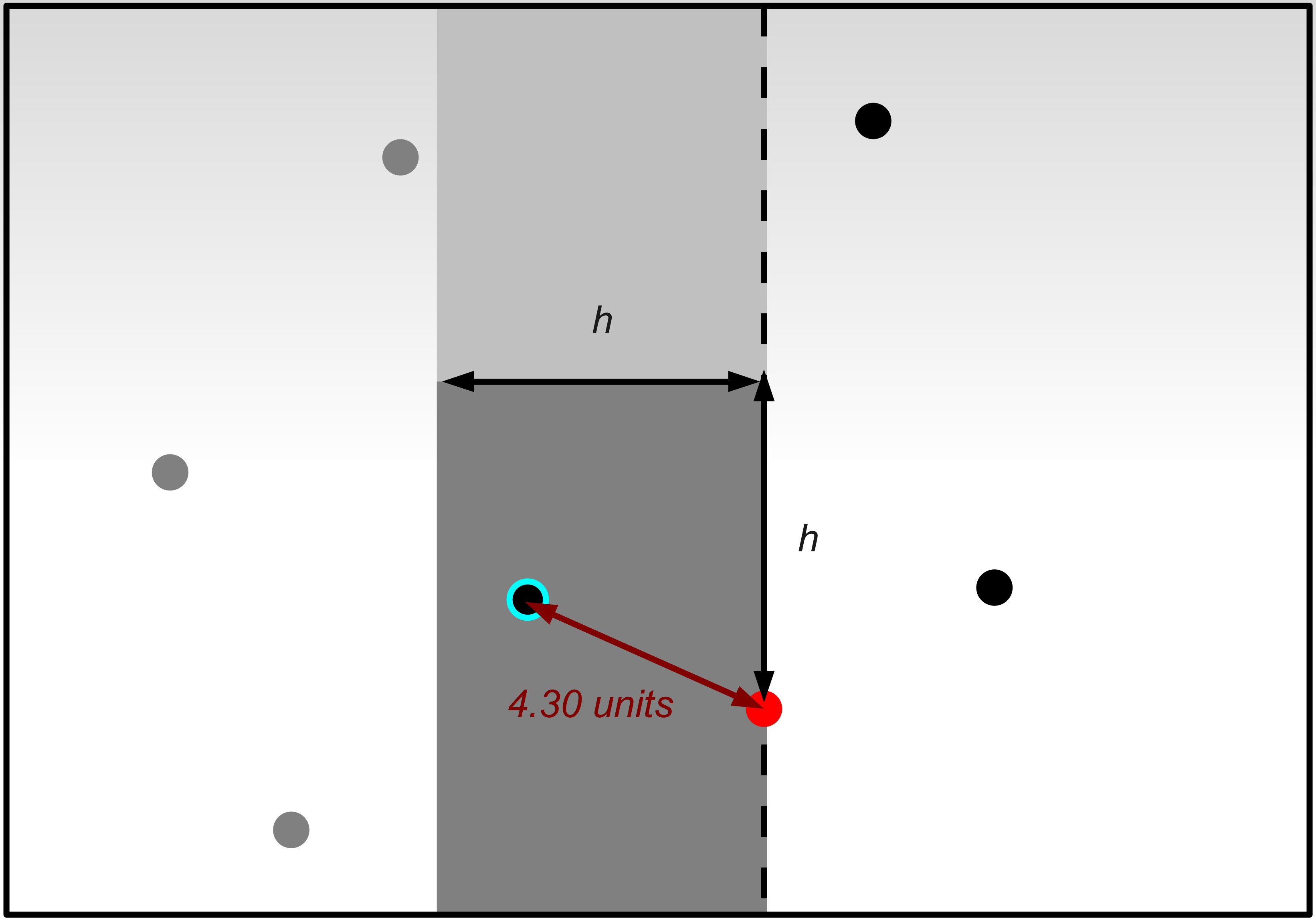
h = Euclidian distance between point 1 and 2 = 6.23 units
No distance less than 6.23 units.

Closest Pair Algorithm



h = Euclidian distance between point 1 and 2 = 6.23 units
There are two points that are closer than the current value of *h*! Change *h* to 5.45.

Closest Pair Algorithm



h = Euclidian distance between point 1 and 2 = 5.45 units
Change *h* to 4.30 units.

Closest Pair Algorithm

- C++ implementation (with STL)

```cpp
#include <stdio.h>
#include <set>
#include <algorithm>
#include <cmath>
using namespace std;
#define px second
#define py first
typedef pair<long long, long long> pairll;
int n;
pairll pnts [100000];
set<pairll> box;
double best;
int compx(pairll a, pairll b) { return a.px<b.px; }
int main () {
        scanf("%d", &n);
        for (int i=0;i<n;++i) scanf("%lld %lld", &pnts[i].px, &pnts[i].py);
        sort(pnts, pnts+n, compx);
        best = 1500000000; // INF
        box.insert(pnts[0]);
        int left = 0;
        for (int i=1;i<n;++i) {
                while (left<i && pnts[i].px-pnts[left].px > best) box.erase(pnts[left++]);
                for (typeof(box.begin()) it=box.lower_bound(make_pair(pnts[i].py-best, pnts[i].px-best));
                                                        it! =box.end() && pnts[i].py+best>=it->py; it++)
                        best = min(best, sqrt(pow(pnts[i].py - it->py, 2.0)+pow(pnts[i].px - it->px, 2.0)));
                box.insert(pnts[i]);
        }
        printf("%.2f\n", best);
        return 0;
}
```

Use a balanced binary tree (Set)

Time complexity: O(N log N)

Closest Pair Algorithm

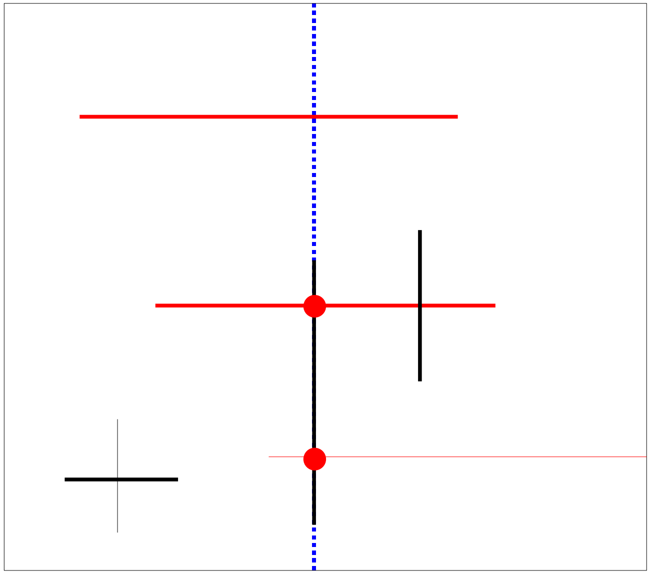
- C++ implementation (with STL) – zoomed in

```
box.insert(pnts[0]);
int left = 0;
for (int i=1;i<n;++i) {
   while (left<i && pnts[i].px-pnts[left].px > best)
        box.erase(pnts[left++]);
   for (typeof(box.begin()) it=
        box.lower_bound(make_pair(pnts[i].py-best, pnts[i].px-best));
        it!=box.end() && pnts[i].py+best>=it->py; it++)
           best = min(best, sqrt(pow(pnts[i].py – it->py,2.0)+
                                 pow(pnts[i].px - it->px, 2.0)));
        box.insert(pnts[i]);
}
printf("%.2f\n", best);
```

Time complexity: O(N log N)

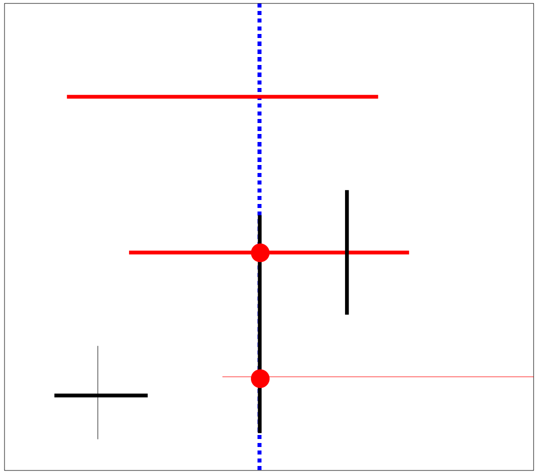
Line segment intersections (HV)

- Problem: given a set S of N closed segments in the plane, report all intersection points among the segment in S
- First consider the problem with only horizontal and vertical line segments
- Brute force: $O(N^2)$ time
  – too slow



Line segment intersections (HV)

- Use events: start of horizontal line, end of horizontal line and vertical line.
- Set contains all horizontal lines cut by the sweep line (sorted by Y). Indicated as red lines on diagram.
- Horizontal line event: add/remove line from set.
- Vertical line event: get all horizontal lines it cuts (get range from set). Indicated as red dots on diagram.
- Use balanced binary tree (C++ set) – guarantee O(log N) for operations.

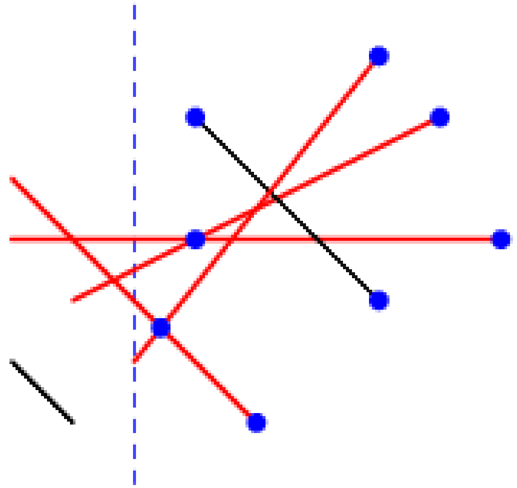


Line segment intersections (HV)

```
// <Headers, structs, declarations, etc.>
// type=0: Starting point of horizontal line
// type=1: Ending point of horizontal line
int main () {
   // <Input>
   sort(events, events+e);                      // Sort events by X-coordinate
   for (int i=0;i<e;++i) {
      event c = events[i];                      // c: current event
      if (c.type==0) s.insert(c.p1);            // Add starting point to set
      else if (c.type==1) s.erase(c.p2);        // Remove ending point from set
      else {
         for (typeof(s.begin()) it=s.lower_bound(point(-1, c.p1.y));
            it!=s.end() && it->y<=c.p2.y; it++) // Range search
              printf("%d, %d\n", events[i].p1.x, it->y);
         }
   }
   return 0;
}
```

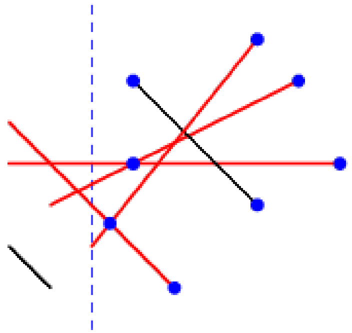
Line segment intersections

- More general case: lines don't have to be horizontal or vertical.
- Lines change places when they intersect.
- Use priority queue to handle events.
- Events also sorted by X.
- Events in priority queue:
  - end-points of line-segments
  - intersection points of adjacent elements.
- Set contains segments that are currently intersecting with the sweep line.



Line segment intersections

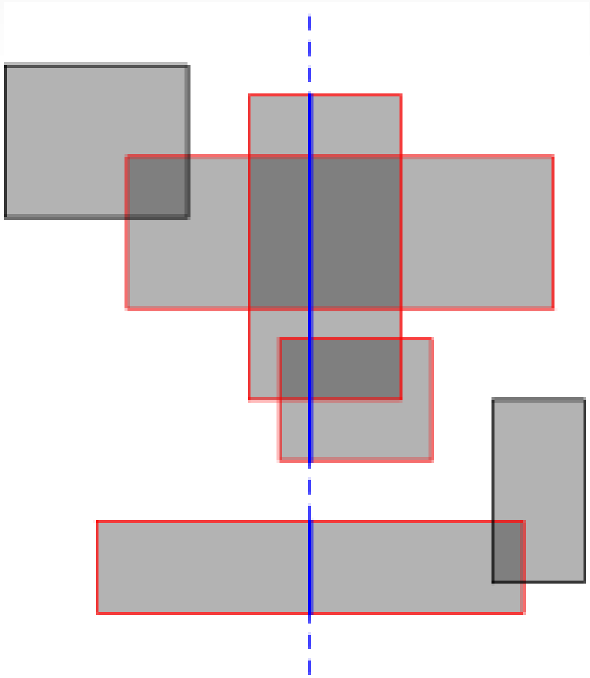
- At a starting point of a line segment:
  - Insert segment into set.
  - Neigbours are no longer adjacent. Delete their intersection point (if any) from the priority queue if it exists.
  - Compute intersection of this point and its neigbours (if any) and insert into priority queue.
- At an ending point of a line segment:
  - Delete segment from set.
  - Neigbours are now adjacent. Compute their intersection point (if any) and insert into priority queue.
- At an intersection point of two line segments:



  - Output point.
  - Swap position of intersection segments in set.
  - The swapped segments have new neigbours now. Insert / delete intersecting points from priority queue (if needed).

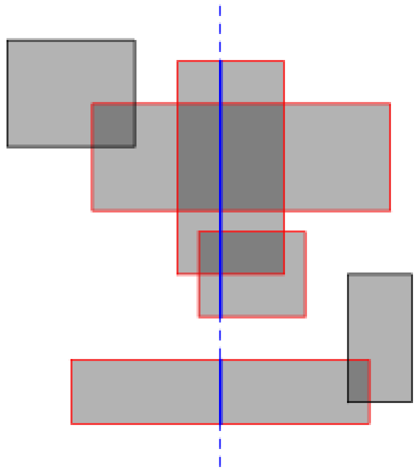
Area of union of rectangles

- Calculate area of the union of a set of rectangles.
- Again work with events (sorted by x) and a set (sorted by y).
- Events:
  - Left edge
  - Right edge
- Set contains all the rectangles the sweep line is crossing.



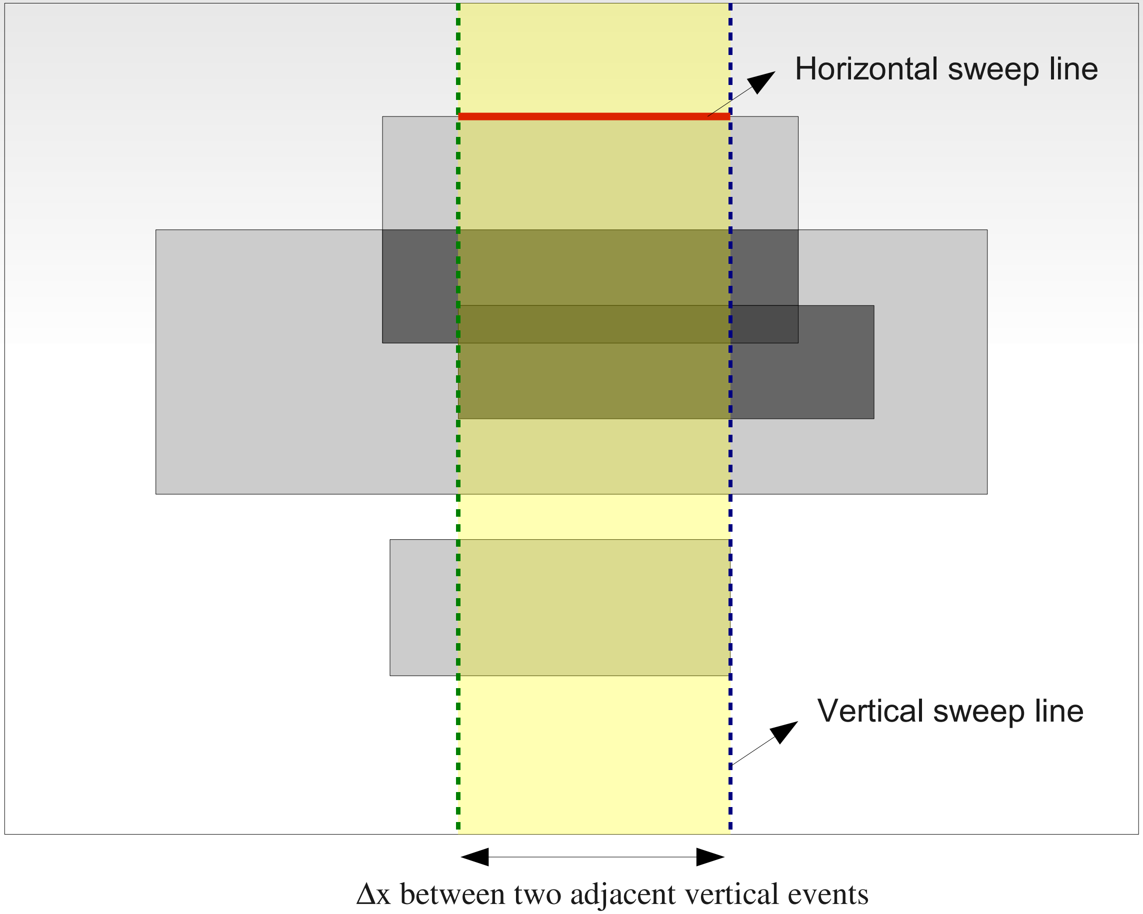
Area of union of rectangles

- Know x-distance (Δx) between two adjacent events.
- Multiply it by the cut length of the sweep line (Δy) to get the total area of rectangles between the two events.
- Do this by running the same algorithm rotated 90 degrees. (Horizontal sweep line running from top to bottom)
  - Use only rectangles in the active set
  - Event: Horizontal edges.
  - Use a counter that indicates how many rectangles are overlapping at the current point.
  - Cut lengths are between two events where the count is zero.



Area of union of rectangles

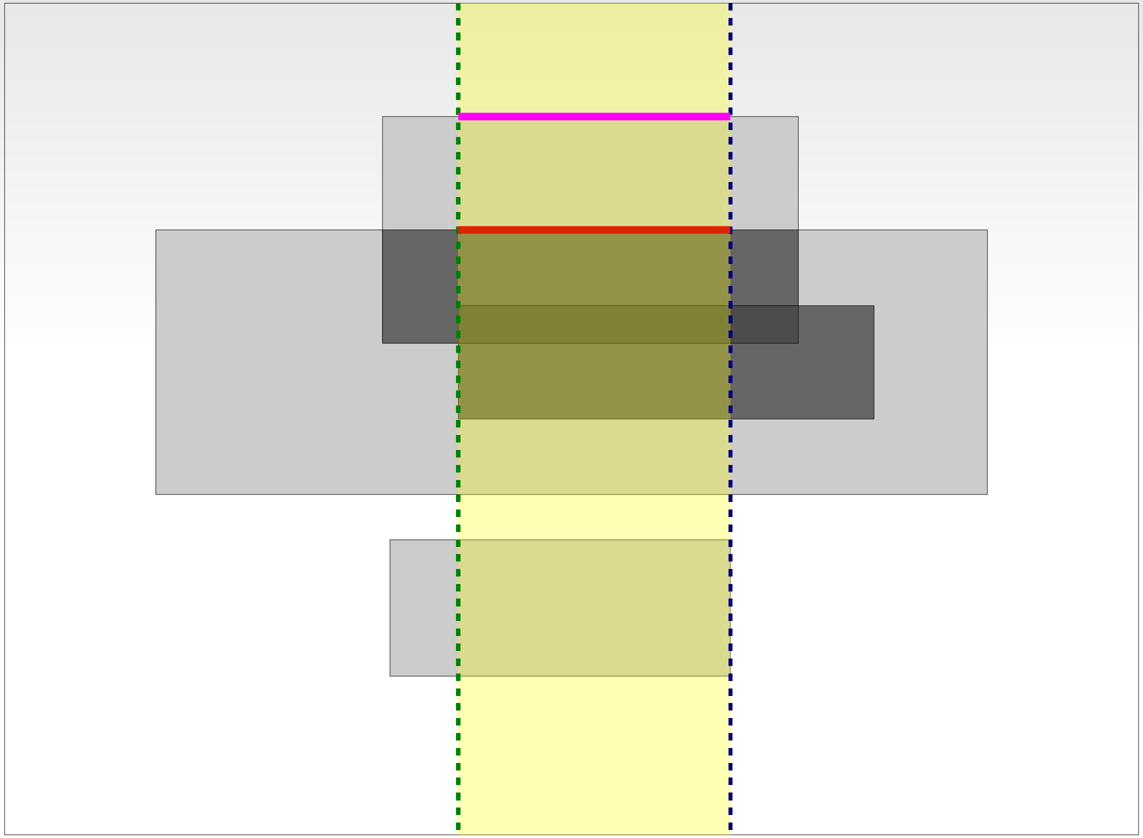
Example of inner loop



Count: 1

Area of union of rectangles

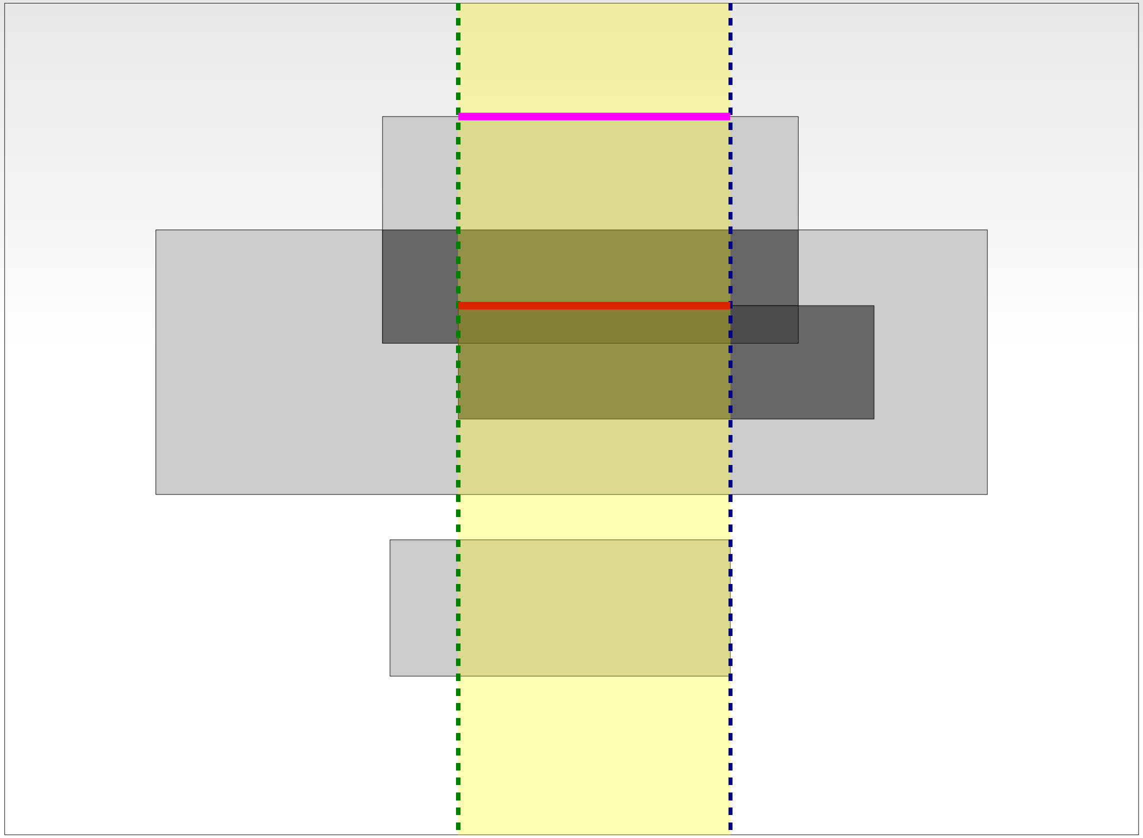
Example of inner loop



Count: 2

Area of union of rectangles

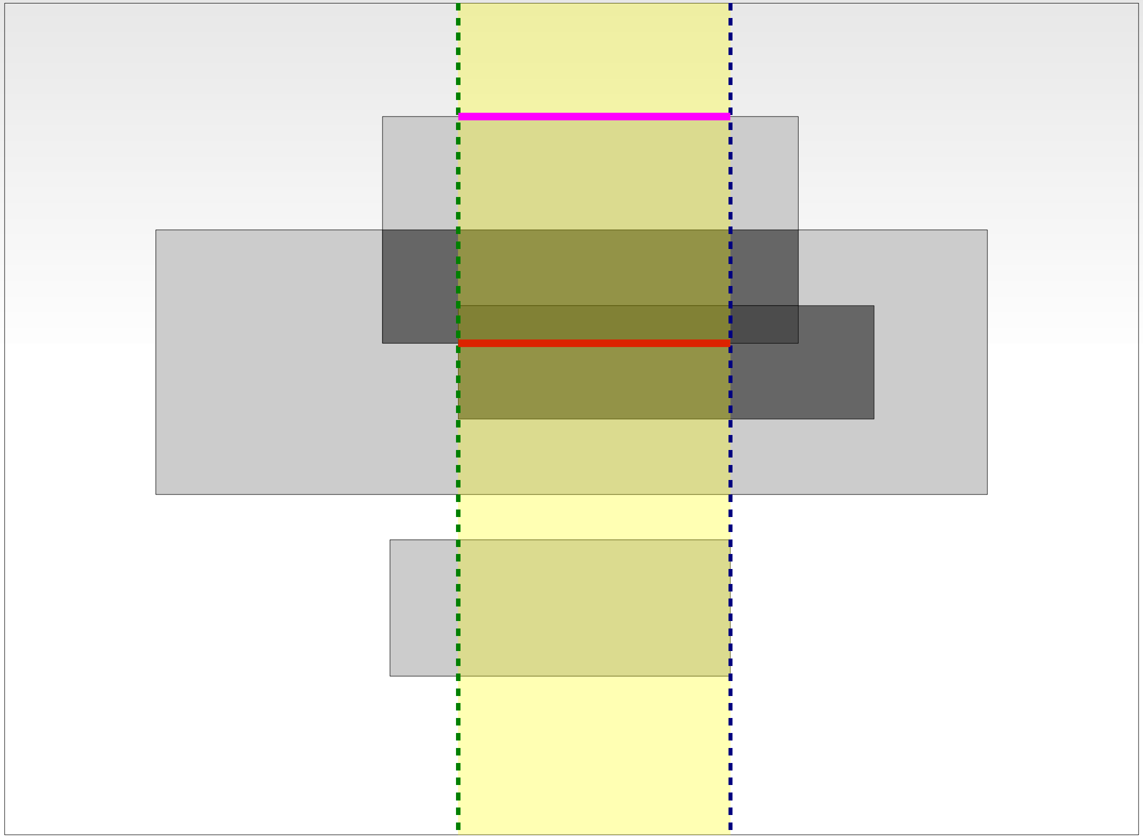
Example of inner loop



Count: 3

Area of union of rectangles

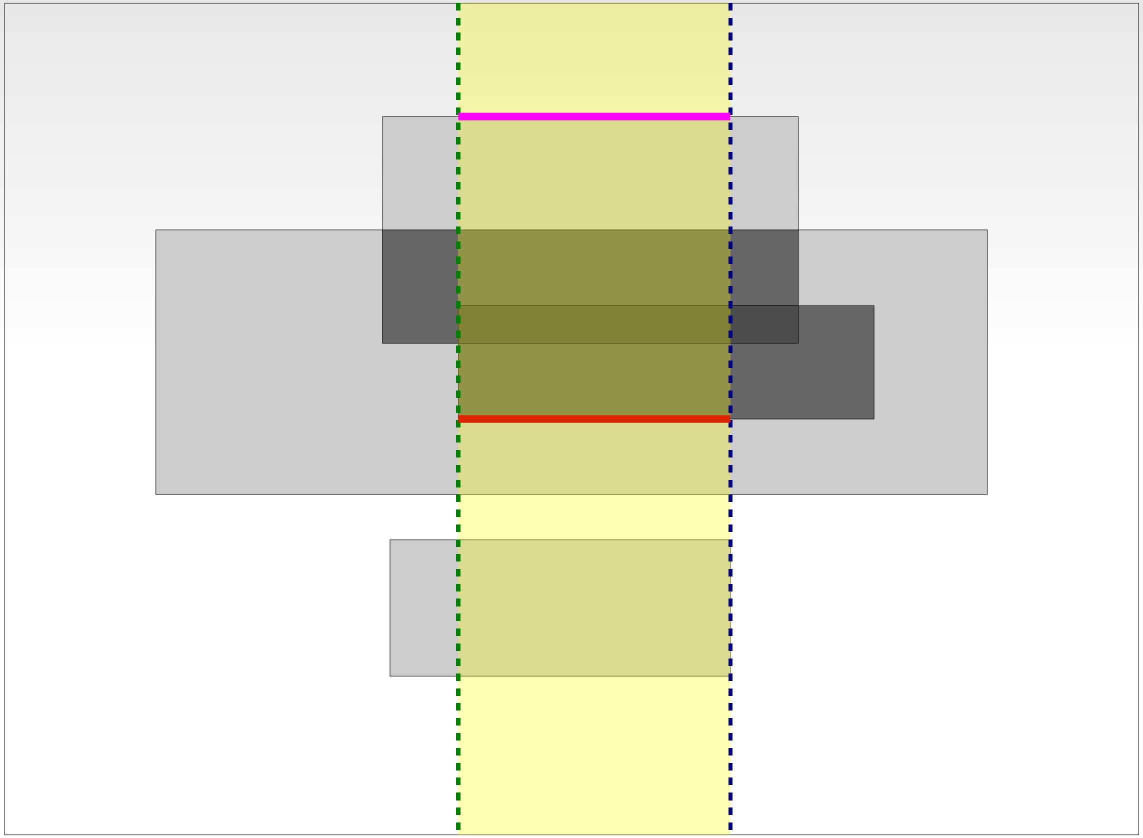
Example of inner loop



Count: 2

Area of union of rectangles

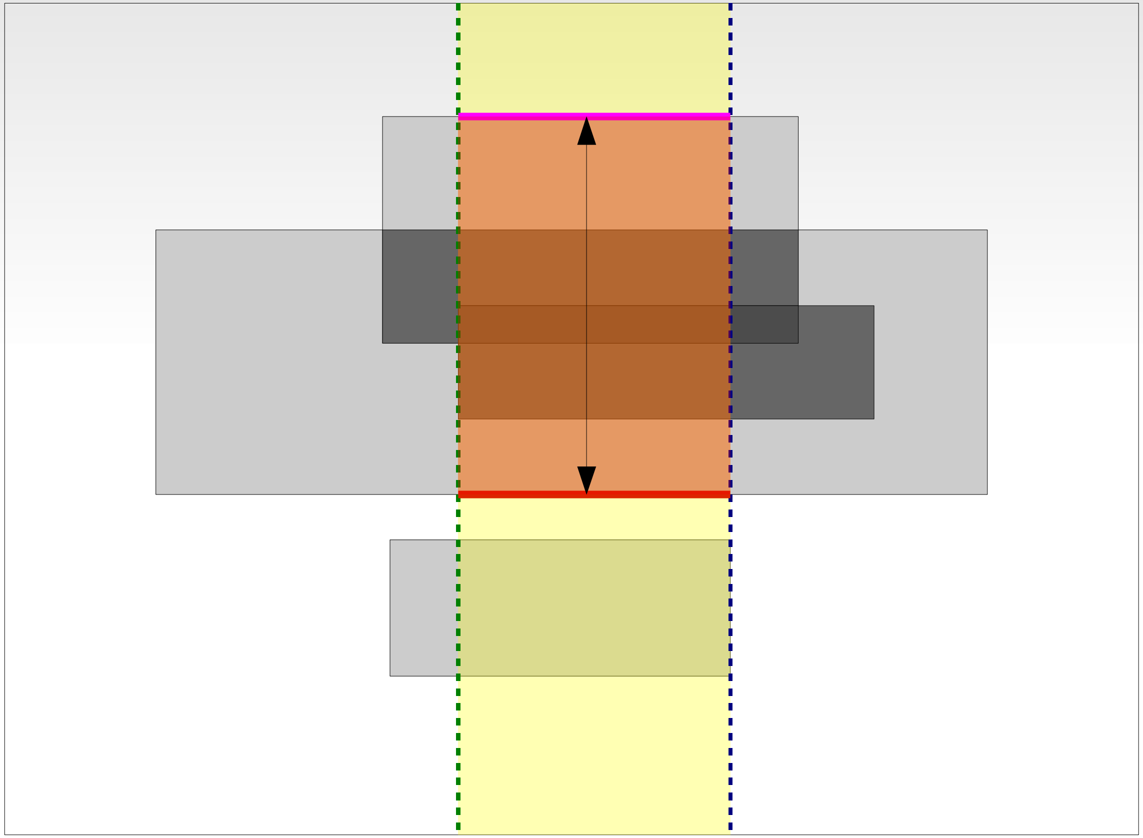
Example of inner loop



Count: 1

Area of union of rectangles

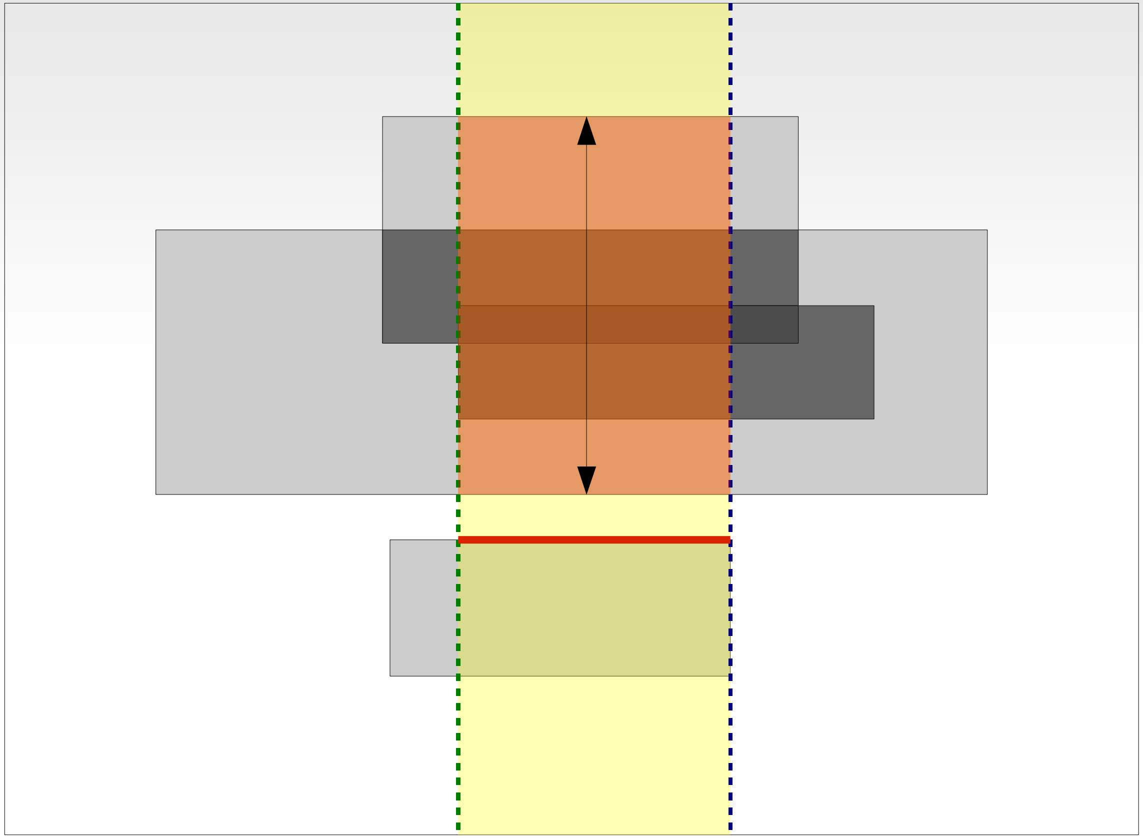
Example of inner loop



Count: 0

Area of union of rectangles

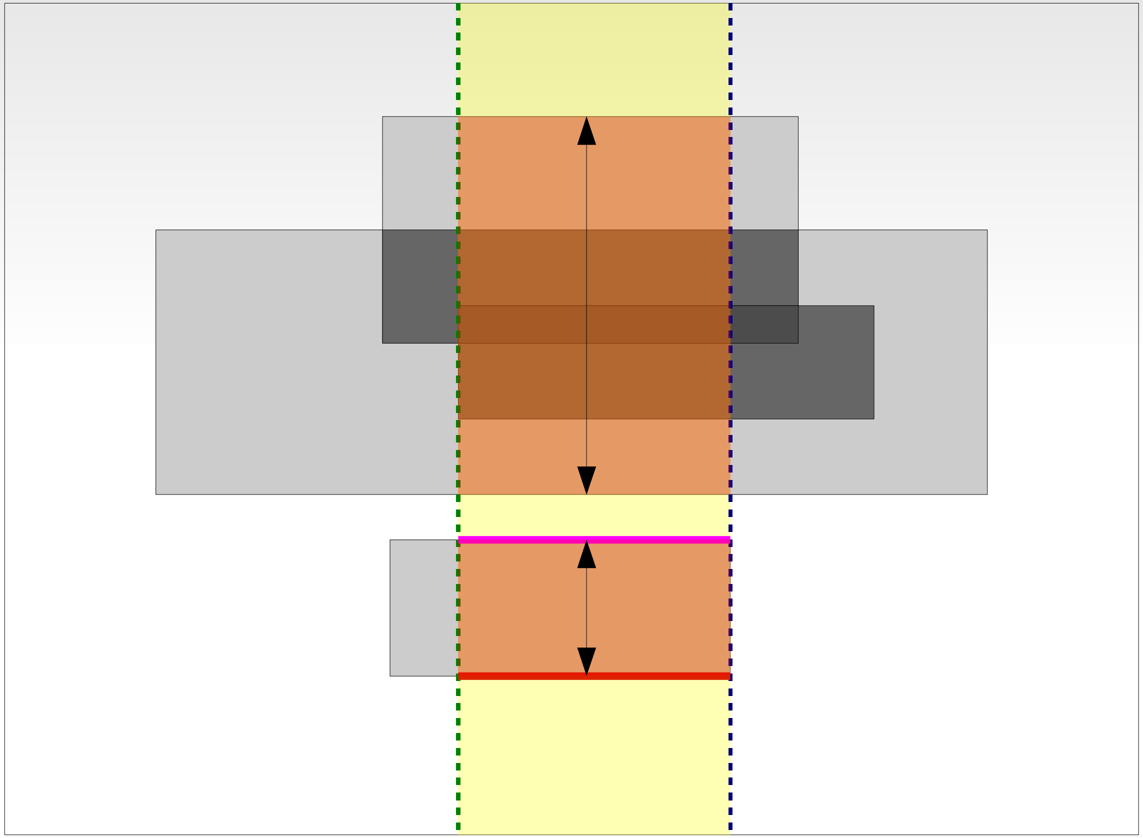
Example of inner loop



Count: 1

Area of union of rectangles

Example of inner loop



Count: 0

Area of union of rectangles

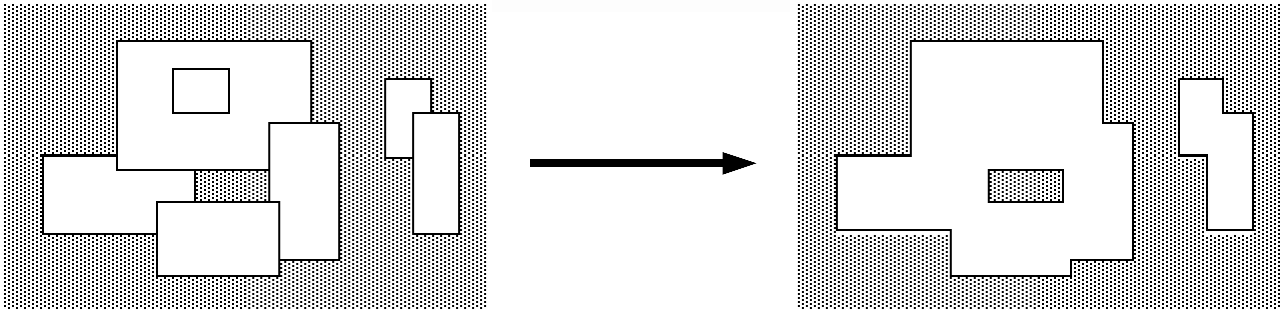
Source code (C++) can be seen on the handout.

Area of union of rectangles

- Run time: $O(N^2)$ with boolean array in the place of a balanced binary tree (pre-sort the set of horizontal edges).
- Can be reduced to $O(N \log N)$ with a binary tree manipulation.

IOI '98: Pictures

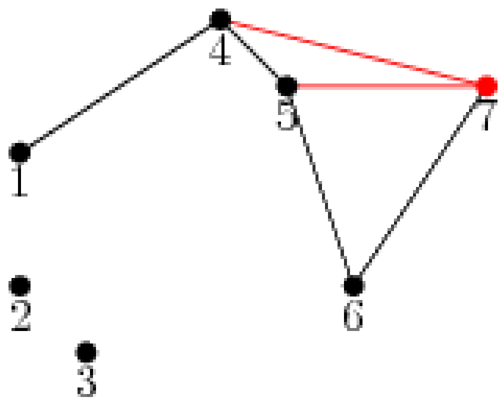
- Given: A number of rectangles. (0<=N<5000)
- Calculate the perimeter (length of the boundary of the union of all rectangles)



- Use basically the same algorithm
- Horizontal boundaries: where count is zero in inner loop.

Convex hull with sweep line

- Graham scan: Sort by angle – is expensive and can get numeric errors. (Can use C++ complex library)
- Simpler algorithm: Andrew's Algorithm
- Sort by X and use a sweep line!
- Upper hull: Start at point with minimum X coordinate and move right. When the last three points aren't convex, delete the second-last point. Repeat until the last three points form a convex triangle.
- Sweep line algorithm runs in O(N)
- O(N log N) – (points are sorted)



Questions?

?

```
presentation.end();
```